



Automatic e-assessment of mathematical word problems

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CAA in mathematics

Computer aided assessment (CAA) of mathematics.

- Mathematical modelling and word problems.
- The STACK CAA system.

Two strands of mathematical activity

(1) The use of routine techniques.

- recognition
- reduction to standard form
- accuracy

Direct instruction

(2) Problem solving.

- novelty
- creativity
- struggle ...
- ... satisfaction?

“discovery”, “investigation” etc.

Mathematical modelling

Transforms

Problem \rightarrow standard system.

... to which routine techniques can be applied.

Solutions \rightarrow Problem.

Classical Newtonian Mechanics

Modelling often taught with *mechanics*.

E.g. vertical oscillations of the spring & mass.

Lots of assumptions/approximations:

- Point mass, massless spring.
- Perfect spring: Hooke's law.
- Small relative to the Earth (gravity constant and down).
- Rigid support (o/w coupled oscillators).
- No air resistance.
- Constant parameters, e.g. temperature variation.

etc.

The model

$$\ddot{y}(t) = -\omega^2 y(t), \quad + \text{initial conditions.}$$

This gives *simple harmonic motion*, $y(t) = a \sin(\omega t + \rho)$.

What is the role of the horizontal component?

- Vertical spring: SHM
- Horizontal pendulum: also SHM!

Projectile motion is de-coupled....

Mechanics is hard!

... not just the calculus ...

... too late to learn modelling as well ...

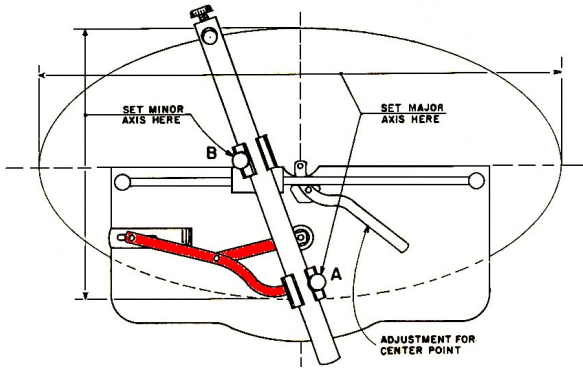
Word problems

Word problems are an early form of modelling.

Example: the cat on the ladder.

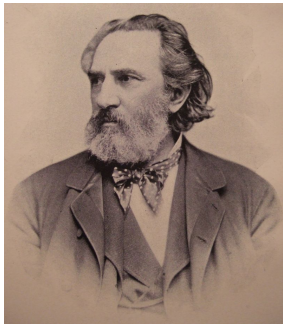
A ladder standing on a smooth floor against a wall slides down to the floor. Along what curve does a cat sitting in the middle of the ladder move?

Asside: grasshopper linkage



Kinematics

The geometry of machine movement.



Franz Reuleaux (1829–1905)

- Kinematics of Machinery, (1876)
- The Constructor, (1904)

Word problems: algebraic versions

Write an equation for the following statement: “*There are six times as many students as professors at this university*”. Use S for the number of students and P for the number of professors.

Word problems: algebraic versions

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When this was given to 150 calculus level students, 37% answered incorrectly with $6S = P$ accounting for two thirds of all errors, *Clement et. al. 1981*.

Polya: Mathematical discovery

I hope that I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary schools is to teach the setting up of equations to solve word problems. Yet there is a strong argument in favor of this opinion. In solving a word problem by setting up equations, the student translates a real situation into mathematical terms; he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out.

[...]

And so the future engineer, when he learns in the secondary school to set up equations to solve “word problems” has a first taste of, and has an opportunity to acquire the attitude essential to, his principal professional use of mathematics.

SEFI, the Société Européenne pour la Formation des Ingénieurs,
2002

The ability to formulate a mathematical model of a given physical situation, to solve the model, interpret the solution and refine the model is a key aspect of the mathematical development of an engineer.

Word exercises

Mayer 1981: a comprehensive classification of 1097 high-school algebra story problems.

Chris takes 10 hours to walk up a mountain and back down by the same route. He averages 2km per hour on the way up and 3km per hour on the way down. How far was it from the base to the top of the mountain?

1. More complex than an arithmetic question.
2. Requires a numerical answer not an equation.
3. Stated in words with a story line.

Classified into major *families* based on their *source formulae*.

E.g. distance, rate and time: $d = r \times t$, or $d = v \times t$.

Story problems contains only three kinds of proposition.

1. Values are assigned to variables,
e.g. time taken to walk up = 10.
2. Relationships between variables,
e.g. distance up = distance back.
3. Identification of the unknown or goal,
e.g. GOAL = distance up.

Within families are *categories*, which are further refined into *templates*.

Problems belong to the same template if they share the same story line and same list of propositions, regardless of the actual values assigned to each variable, the actual relation assigned to a pair of variables, or which variable is assigned to the unknown. *Mayer, 1981, pg 145.*

Our example problem is an instance of

- the *round-trip* template
- a sub-category of the *distance—time* category
- of the *rate* family

Our source formulae

d_1 = distance 1 (up), GOAL

t_1 = time 1 (up)

v_1 = rate 1 (up), 2km/h

etc.

$$d_1 = v_1 \times t_1$$

$$d_2 = v_2 \times t_2$$

$$d_1 = d_2$$

$$d_t = d_1 + d_2$$

$$t_t = t_1 + t_2$$

Word exercises

Traditional “word exercises” have elements of modelling.

1. Choices, e.g. coordinates
2. Often no approximations.
3. Correct answers.
4. Multiple steps.
5. Discussion on assumptions.
6. Algebra, geometry, trig. NOT calculus.

Introduction to STACK

STACK: a computer aided assessment system for mathematics.

The focus is on

- Evaluating students' answers.
(Not MCQ or similar)
- Generating structured random questions.
- Feedback, based upon *properties* of the student's answer.

System demonstration

STACK <http://www.stack.bham.ac.uk>

Word exercises in STACK

In a railway journey of 90 kilometres an increase of 5 kilometres per hour in the speed decreases the time taken by 15 minutes. What is the speed?

One system of equations:

$$\{90 = vt, 90 = (v + 5)(t - 1/4)\}. \quad (1)$$

Goal is to find v .

Eliminate t from the second equation: $v^2 + 5v - 1800 = 0$

- also belongs to the distance, rate (v) and time family,
- mix of units of time between hours and minutes
- use judgement to choose quadratic root.

Buchberger's Algorithm

Polynomials can be written in a *canonical form* such as

$$ax^2 + bx + c.$$

Equations also have such a form which can be calculated by using *Buchberger's Algorithm*

- For a linear system = Gaussian elimination.
- For single variable polynomials = Euclid's algorithm.

In our example the *reduced* Gröbner basis is

$$\{-20t + v + 5 = 0, v^2 + 5v - 1800 = 0\}. \quad (2)$$

Conclusion

The equivalence of systems of equations can be established automatically.

This enables CAA of modelling in traditional algebra story problems.